J. Fluid Mech. (1967), *vol.* 29, *part* 2, *pp.* 401–413
Printed in Great Britain

Magnetohydrodynamic flow at a rear stagnation point

By S. LEIBOVICH
Department of Thermal Engineering, Cornell University, Ithaca, New York

(Received 5 September 1966)

An exact rear stagnation point solution is sought for a viscous, incompressible conducting fluid in the presence of a magnetic field. It is found that a steady solution exists only if $\mathcal{N} \geqslant 2$, where $\mathcal{N} = \sigma B^2/\rho\alpha$ is the interaction parameter of the flow based upon the normal component of the magnetic field at the wall. Here α is a positive rate of strain, which, for a finite body with length l and velocity U_0, is of the order of U_0/l.

The steady solution is found, and from its existence it is inferred that separation of the viscous boundary layer does not begin at the rear stagnation point when $\mathcal{N} \geqslant 2$, and that such separation can be prevented. This supports theoretical work by Moreau (1964) and experimental work of Tsinober, Shtern & Shcherbinin (1963).

When $\mathcal{N} < 2$, the flow is necessarily unsteady, and in this case an asymptotic analysis (as $t \to \infty$) similar to that of Proudman & Johnson (1962) is undertaken. For $\mathcal{N} < 1$, the magnetic and non-magnetic flows are qualitatively alike, in that there is a growing inviscid region dominated by eddies, and an ultimately steady layer at the wall representing a viscous *forward* stagnation point flow.

For $1 < \mathcal{N} < 2$, the inviscid region again grows with time, but no eddies appear. It is thus suggested that for this range of $\mathcal{N}$ separation occurs without reversed flow.

1. Introduction

In plane flow of an incompressible, viscous, electrically conducting fluid over a solid body, the presence of a normal magnetic field at the surface has the effect of alleviating an adverse pressure gradient. One might expect that separation of the boundary layer would be delayed as a result. Indeed, since the degree to which the unfavourable pressure gradient is alleviated depends on the strength of the magnetic field, it is conceivable that separation could be completely suppressed.

There is evidence which suggests that this is in fact the case. Experiments performed by Tsinober *et al.* (1963) on the flow of mercury past circular cylinders indicate that the position of the separation point is a function of the interaction parameter $N = M^2/Re$ (M = Hartmann number, Re = Reynolds number). As N increases from zero, the separation point moves downstream. Depending upon the conductivity of the cylinders, separation was reported to be completely inhibited for values of N between 1 and 2. (See, however, the discussion to follow in §8.)

So far as the boundary-layer equations, and hence the possibility of separation, are concerned, it is the strength of the component of magnetic field normal to the body surface which is critical. In a theoretical analysis of flow past a cylinder, therefore, the magnetic effect is exhibited most clearly by assuming that a radial magnetic field has been established. (The way such a field may be created in practice has been described by Heiser & Shercliff (1965).)

Under these assumptions, Moreau (1964) showed that separation is prevented for $N \geqslant 2$. He used an approximate method of Meksyn's and assumed that the external flow is unaffected by the magnetic field.

A detailed numerical solution of the boundary-layer equation for the identical problem has been given by Fucks, Fischer & Uhlenbusch (1964) for $N = 0, 0{\cdot}3, 1$. They found that the angle of separation increased from 110° at $N = 0$ to 144° at $N = 1$. The case $N = 10$ was also calculated and gave a separation angle of 169°, and the authors suggest that separation is prevented only for $N = \infty$. This last result disagrees with the experimental findings of Tsinober *et al.*, and the theoretical work of Moreau. However, the authors state that their method loses accuracy as N increases, and that for $N > 1$ the results are not reliable.

This paper presents further evidence that the steady boundary-layer remains attached for N, based upon the normal component of magnetic field, greater than or equal to 2. It is assumed that the magnetic Prandtl number $\mu\sigma\nu \ll 1$. Then the induced magnetic field in the viscous boundary layer is negligible, even for large magnetic Reynolds numbers (R_M), as Sears (1961) has shown.

In contrast to the previous work on this problem, we do not need to deal with special geometries, such as circular cylinders.† The analysis holds quite generally for bodies blunt enough to be approximated by a plane wall at the rear stagnation point (herein abbreviated as R.S.P.). Furthermore, the magnetic field need not be normal at the surface.

The approach involves the assumption that the boundary layer remains attached, and the flow at the R.S.P. is investigated. Equations similar to the 'backwards' boundary layers discussed by Goldstein (1965) are involved. No solution to these equations exists for $N < 2$.‡ In the non-magnetic case, this is a familiar result, and indicates that the boundary layers must separate before reaching the R.S.P. An illuminating discussion of this point has been given by Proudman & Johnson (1962).

For $N \geqslant 2$, we present numerical results for an exact steady R.S.P. solution at a plane wall. In particular, numerical results for $N = 2{\cdot}0$, $2{\cdot}5$, 5, $7{\cdot}5$ and 10 are given in §5. This solution decays algebraically in the distance normal to the wall. In the light of recent work by Brown & Stewartson (1965), algebraic behaviour is permissible in such a limiting solution, and we consider that the solution supports Moreau's work, and provides further theoretical foundation for the experimental evidence.

† Moreau also treats the flat plate for various external flows.

‡ Hardy's (1939) non-existence proof may easily be adapted to our problem, and shows rigorously that this is the case.

Proudman & Johnson (1962) have discussed R.S.P. flow for $N = 0$, for which the flow is necessarily unsteady. In §7, we repeat their analysis for the magnetic case $0 \leqslant N < 2$. In particular, we assume that a steady R.S.P. flow has been established, with $N \geqslant 2$. At time $t = 0$, the magnetic field is reduced so that N assumes a value less than 2, and the flow must then become unsteady.

Although, for completeness, we have extracted the essential points of Proudman & Johnson's arguments, the interested reader is advised to familiarize himself with their paper, the clarity of which can hardly be improved.

Results analogous to those found by Proudman & Johnson were recovered for $N < 1$. More explicitly, it was found that the R.S.P. moves out to infinity in the direction normal to the wall. The flow near the wall is ultimately that for a *forward* stagnation point. Inviscid eddies, which grow in thickness exponentially with time, bridge the gap between this ultimately steady viscous layer and the R.S.P. flow far away.

This picture is qualitatively different if $1 < N < 2$. Again an inviscid region grows exponentially in thickness with time, and bridges the gap between the retreating R.S.P. and an ultimately steady viscous layer at the wall. Now, however, the flow in the inviscid region does not change direction, there is no reversed flow, and so the flow there is no longer characterized by eddies.

In fact, the normal velocity attained at the outer edge of the viscous layer is $N-1$ (for all $N < 2$), which is away from the wall for $N > 1$. The outer boundary condition on the viscous layer is hence that of a R.S.P. flow with reduced normal velocity. As will be shown in §4, a steady viscous R.S.P. solution can exist at the wall with a normal velocity equal to $N-1$ at infinity. Therefore, for all N, a steady viscous layer forms at the wall with this model. When $N < 1$, fluid moves towards the wall, and, when $N > 1$, it moves away from it. It is the character of the inviscid flow which changes dramatically from steady to unsteady as N passes through 2.

It is remarkable that, for $N > 1$, the skin friction at the wall is non-zero and positive. Hence, if $1 < N < 2$, and the Proudman–Johnson model of the unsteady flow is appropriate, a phenomenon which may be identified as separation occurs without reversed flow.

This is a novel point, and, if observable experimentally, could make the use of magnetic fields a valuable tool in studying the process of separation.

2. Formulation of the problem

We look for an exact solution to the problem of two-dimensional flow past an infinite plane wall. The co-ordinate x is measured along the wall from the stagnation point, and the y co-ordinate is normal to the plane, the fluid occupying the half-space $y > 0$. A magnetic field is imposed at the boundary at an arbitrary angle, θ, to the x-axis.

The equations of motion for the flow of an incompressible conducting fluid are

$$\left.\begin{aligned} (\partial \mathbf{q}/\partial t) + \mathbf{q}\,.\,\nabla\mathbf{q} + (1/\rho)\,\nabla p &= (\sigma/\rho)\,[\mathbf{E}\times\mathbf{B} + (\mathbf{q}\,.\,\mathbf{B})\,\mathbf{B} - \mathbf{q}B^2] + \nu\nabla^2\mathbf{q},\\ \nabla^2\mathbf{B} &= \mu\sigma[\mathbf{q}\,.\,\nabla\mathbf{B} - \mathbf{B}\,.\,\nabla\mathbf{q} + (\partial\mathbf{B}/\partial t)],\\ \nabla\,.\,\mathbf{q} &= 0,\\ \operatorname{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t. \end{aligned}\right\} \quad (1)$$

Here $\mathbf{q} = (u, v)$ is the velocity vector, and the symbols in (1) have their usual meaning.

We assume that the streamfunction Ψ is of the form

$$\Psi = -\alpha x \mathscr{F}(y,t) + \alpha \mathscr{G}(y,t), \tag{2}$$

where α is a constant, positive rate of strain, $\mathscr{F}(y,t)$ has dimensions of length, and $\mathscr{G}(y,t)$ has dimensions of length squared. The natural length scale of the problem is the viscous one, $(\nu/\alpha)^{\frac{1}{2}}$, the time scale is evidently measured by α^{-1}, so that velocity is compared to $(\alpha\nu)^{\frac{1}{2}}$, and pressure to $\rho\alpha\nu$. Introducing these normalizing factors into (1), with $\mathbf{B}$ scaled on a characteristic value B_0 (say), and the electric field referred to $B_0(\alpha\nu)^{\frac{1}{2}}$, (1) may be written

$$\left.\begin{aligned} (\partial \mathbf{q}/\partial t) + \mathbf{q}.\nabla\mathbf{q} + \nabla p &= N[\mathbf{E}\times\mathbf{B} + (\mathbf{q}.\mathbf{B})\mathbf{B} - B^2\mathbf{q}] + \nabla^2\mathbf{q}, \\ \nabla^2\mathbf{B} &= \mu\sigma\nu[\mathbf{q}.\nabla\mathbf{B} - \mathbf{B}.\nabla\mathbf{q} + (\partial\mathbf{B}/\partial t)], \\ \nabla.\mathbf{q} &= 0, \\ \operatorname{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t, \quad \psi = -xF(y,t) + 2\cot\theta G(y,t).\dagger \end{aligned}\right\} \tag{3}$$

All quantities appearing in (3) are dimensionless, ψ being the dimensionless streamfunction, and F and G dimensionless functions bearing an obvious relationship to $\mathscr{F}$ and $\mathscr{G}$.

In (3),

$$N = \sigma B_0^2/\rho\alpha \tag{4}$$

is the interaction parameter of the flow, while

$$\mu\sigma\nu = P_{rM} \tag{5}$$

is the magnetic Reynolds number, based upon the above scalings, also known as P_{rM}, the magnetic Prandtl number (cf. Sears 1961). (In terms of a finite body, $\mu\sigma\nu = R_M/Re$ where the two parameters are computed on the scale of the body.)

We assume that $\mu\sigma\nu \ll 1$, but $N = O(1)$. Then, to order P_{rM}, $\mathbf{B}$ is a constant vector, so

$$\mathbf{B} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \tag{6}$$

and, as a consequence, $\mathbf{E}$ is also a constant vector $= E_0\mathbf{k}$ (say), where $\mathbf{k}$ is the unit vector in the z-direction.

It is required to find a solution to (3) which satisfies the no-slip conditions

$$u = \frac{\partial\psi}{\partial y} = 0; \quad v = -\frac{\partial\psi}{\partial x} = 0, \quad \text{at} \quad y = 0, \tag{7}$$

and which approaches the steady inviscid stagnation point solution of the equations of motion

$$\psi = -y(x - \cot\theta y) \tag{3a}$$

as $y \to \infty$, for all t. The steady case is dealt with first, in §§3–6, and the unsteady case is further formulated and discussed in §7.

From (3a), it is seen that the fluid must leave the vicinity of the stagnation point in the direction of the magnetic field.

† The factor 2 cot θ is inserted for convenience.

To be consistent with the assumed form of the streamfunction, the pressure must be of the form

$$p = \tfrac{1}{2}Px^2 + H(y,t) + xK(y,t) + NE_0(y\cos\theta - x\sin\theta), \tag{8}$$

where P is a constant, and must approach the inviscid pressure appropriate to (3*a*), i.e.

$$p \sim -\tfrac{1}{2}(x^2+y^2-N\sin^2\theta x^2 - N\cos^2\theta y^2) + NE_0(y\cos\theta - x\sin\theta) - N\sin\theta\cos\theta xy, \tag{8a}$$

as $y\to\infty$. Thus $P = N\sin^2\theta - 1$, and (3) may be written

$$F_{yt} + FF_{yy} - F_y^2 + N\sin^2\theta F_y - N\sin^2\theta + 1 = F_{yyy}, \tag{9a}$$

$$G_{yt} + FG_{yy} - F_yG_y + \tfrac{1}{2}\tan\theta K - \tfrac{1}{2}N\sin^2\theta F + N\sin^2\theta G_y = G_{yyy}, \tag{9b}$$

$$K_y + N\sin\theta\cos\theta F_y = 0, \tag{9c}$$

$$F_{yt} + FF_y + H_y - N[2\cos^2\theta G_y - F\cos^2\theta] = F_{yy}. \tag{9d}$$

Here subscripts denote derivatives. The last equation determines H, given the flow field, while (9*c*) may be integrated with result

$$K = -N\sin\theta\cos\theta F + C(t). \tag{9e}$$

In order to satisfy (8*a*), as $y\to\infty$, $C(t) = 0$.

3. The steady viscous flow

In this section, the steady version of equations (9) are treated, so that all flow quantities are considered to be time-independent.

In the analysis for F and G, it is convenient to introduce the interaction parameter based upon the normal component of magnetic field

$$\mathcal{N} = N\sin^2\theta. \tag{10}$$

The results of the previous section show that the steady viscous flow must satisfy the equations:

$$F''' - FF'' + F'^2 - 1 - \mathcal{N}(F'-1) = 0, \tag{11a}$$

$$G''' - FG'' + F'G' + \mathcal{N}(F - G') = 0, \tag{11b}$$

$$H' = F'' - FF' + \mathcal{N}\cot^2\theta[2G' - F], \tag{11c}$$

$$K = -\mathcal{N}\cot\theta F, \tag{11d}$$

with boundary conditions

$$F(0) = F'(0) = G'(0) = 0, \tag{12}$$

$$F'\to 1, \quad G''\to 1 \quad \text{as} \quad y\to\infty.$$

Primes denote derivatives with respect to y.

Once F has been determined, G is found from the linear equation (11*b*). In fact that solution may be written down directly in terms of F. Put

$$G' = F + g; \tag{13}$$

then
$$g(0) = 0, \quad g \to 0 \quad \text{as} \quad y \to \infty,$$
and
$$g'' - Fg' + (F' - \mathcal{N})\, g = -F''. \tag{14}$$

As with the case of magnetohydrodynamic stagnation point flow at a point of attachment (Ludford 1963), and with the analogous non-magnetic case (Glauert 1956),
$$g_1 = F''(y) \tag{15}$$
is one complementary function. A second may be constructed from it, viz.
$$g_2 = F''(y) \int_0^y \left(\exp \int_0^\eta F(t)\, dt\right) \frac{d\eta}{[F''(\eta)]^2}. \tag{16}$$
The solution to the boundary-value problem for g is then
$$g = \int_0^y g_1(y) g_2(\eta) F''(\eta) \exp\left(-\int_0^\eta F\, dt\right) d\eta + \int_y^\infty g_2(y) g_1(\eta) F''(\eta) \exp\left(-\int_0^\eta F\, dt\right) d\eta. \tag{17}$$
Convergence of the integrals is assured by the asymptotic behaviour of F as found below.

To investigate the asymptotic behaviour of (11a), put
$$F = y + f, \tag{18}$$
where $f = o(y)$ as $y \to \infty$, and
$$f''' - yf'' + (2 - \mathcal{N}) f' = ff'' - f'^2, \tag{19}$$
and for large y the terms on the right-hand side are negligible compared with those on the left.

Two independent solutions of (19) for f' are
$$e^{\frac{1}{4}y^2} D_{2-\mathcal{N}}(y), \tag{20}$$
$$e^{\frac{1}{4}y^2} D_{\mathcal{N}-3}(-iy). \tag{21}$$

As $y \to \infty$, (20) is asymptotic to
$$y^{2-\mathcal{N}} \left\{1 - \frac{(2-\mathcal{N})(1-\mathcal{N})}{2y^2} + \dots\right\}$$
and (21) to
$$e^{\frac{1}{2}y^2} y^{\mathcal{N}-3} \{1 + O(y^{-2})\}.$$

The last solution diverges with y, as does (20) if $\mathcal{N} < 2$. However, if $\mathcal{N} \geqslant 2$, a solution may be possible which behaves like
$$F \sim y + A\, e^{\frac{1}{4}y^2} D_{3-\mathcal{N}}(y) \tag{22}$$
for large y.

Results of this kind were found by Goldstein (1965) for similar equations. That no solution exists for $\mathcal{N} < 2$ may be confirmed by a rigorous non-existence proof given by Hardy (1939). To verify that Hardy's proof applies to our (slightly different) equation is a simple matter, and so we do not demonstrate it here. If $\mathcal{N} \geqslant 2$, however, the proof fails. This, and the asymptotic behaviour of the governing equations, suggest that a solution may exist for this case. In fact, rigorous existence and uniqueness results have been obtained by Leibovich (1967) for $\mathcal{N} \geqslant 2$, and will be presented elsewhere. Some numerical results for this case are given in §5.

Before concluding this section, it should be pointed out that the asymptotic analysis can be repeated with a relaxed boundary condition. In particular, it is consistent with equations (11a) to assume that $F' \to \mathcal{K} = \mathcal{N} - 1$ as $y \to \infty$ if $\mathcal{N} < 2$ (but obviously no other $\mathcal{K}$ is acceptable). The asymptotics suggest that there is a solution to this problem for $\mathcal{N} < 2$, and in fact existence and uniqueness can be proved in the same way as in Leibovich (1967). Of course, for $\mathcal{N} = 2$ this problem and the central one posed in (11) and (12) coincide.

It might be thought that there could be a steady, inviscid layer which could provide a transition from the R.S.P. flow with $F' = 1$, and the viscous layer with $F' \to \mathcal{N} - 1$, when $\mathcal{N} < 2$. Such a layer would be a steady counterpart of the unsteady inviscid layer found by Proudman & Johnson (1962) and further discussed in §7 of this paper. The next section, however, shows that no such steady layer can exist, and that therefore if $\mathcal{N} < 2$ there is no steady R.S.P. solution.

4. Steady inviscid stagnation point flow

The steady inviscid flow, that is, the flow far from the wall, satisfies (9) with the right-hand sides and time derivatives set equal to zero. The principal boundary conditions to be satisfied are

$$F_i(0) = G_i'(0) = 0, \tag{23}$$

where the subscript identifies inviscid quantities. The second of these conditions is required since $u = v = 0$ at the inviscid stagnation point $x = y = 0$.

The two equations for the streamfunction are

$$F_i F_i''' - F_i'^2 + \mathcal{N} F_i' = \mathcal{N} - 1, \tag{24a}$$

$$F_i G_i'' - F_i' G_i' + \mathcal{N} G_i' = \mathcal{N} F_i, \tag{24b}$$

and (24a) may be written as

$$zF_i \frac{dz}{dF_i} = z^2 - \mathcal{N} z + \mathcal{N} - 1, \tag{25}$$

where $z = F_i'$.

The solution to (25) is therefore

$$kF_i = \left\{ \frac{(z - \mathcal{N} + 1)^{\mathcal{N}-1}}{z-1} \right\}^{\mathcal{N}-2}, \tag{26}$$

where k is a constant. If $\mathcal{N} < 2$, $k = 0$, since $z = 1$ at $F_i = \infty$. The solution in this case is therefore

$$F_i = y, \tag{27}$$

and so there is no steady inviscid layer which can provide the transition between the viscous layer and an R.S.P. flow at infinity for $\mathcal{N} < 2$.

5. The numerical solution

We assume that $F'' > 0$ throughout,† then $F \geqslant 0$, and $0 \leqslant F' < 1$, and $F''' < 0$ for all y, and then proceed following Coppel (1960). Consequently,

$$FF'' - \tfrac{1}{2}F'^2 \leqslant 0$$

since it is a decreasing function of y, as may be seen by differentiation, and is initially zero. Thus

$$F''' \leqslant 1 - \tfrac{1}{2}F'^2 - \mathcal{N}(1-F'). \tag{28}$$

Put $F' = z$, $F'' = w$, then (28) is equivalent to

$$w\frac{dw}{dz} \leqslant 1 - \tfrac{1}{2}z^2 - \mathcal{N}(1-z). \tag{29}$$

Consider the function $w^*(z)$ for which the equality holds, and $w^*(1) = 0$,

$$w^{*2} = \mathcal{N}(z-1)^2 + \tfrac{1}{3}(1-z^3) + 2(z-1). \tag{30}$$

But $dw^2/dz \leqslant dw^{*2}/dz$, and $w^*(1) = w(1)$; therefore $w^2 \geqslant w^{*2}$ in the interval $0 \leqslant z \leqslant 1$.

Hence

$$F''(0) \geqslant (\mathcal{N} - \tfrac{5}{3})^{\frac{1}{2}}. \tag{31}$$

Similarly,

$$F''' \geqslant 1 - F'^2 - \mathcal{N}(1-F').$$

Again, a w^* may be found, and, since $dw^{*2}/dz \leqslant dw^2/dz$, $w \leqslant w^*$ in $0 \leqslant z \leqslant 1$. In this way we find

$$F''(0) \leqslant (\mathcal{N} - \tfrac{4}{3})^{\frac{1}{2}}. \tag{32}$$

Equations (31) and (32) furnish a valuable aid to the numerical calculations. It should be emphasized, however, that they hold only for solutions in which $F'' > 0$ for all y.

The numerical results are summarized in the table below, which gives $F''(0)$ for $\mathcal{N} = 2{\cdot}0, 2{\cdot}5, 5{\cdot}0, 7{\cdot}5, 10{\cdot}0$:

$\mathcal{N}$	2·0	2·5	5·0	7·5	10·0
$F''(0)$	0·768	1·025	1·874	2·451	2·916

The most interesting feature of this table is that $F''(0)$ does not vanish at the critical value $\mathcal{N} = 2$. Since the flow separates for $\mathcal{N} < 2$, this implies either that $F''(0)$ does not depend continuously on $\mathcal{N}$, or that separation occurs without reversed flow. In §7, it is suggested that the latter is the case.

6. Asymptotic solution for large $\mathcal{N}$

As $\mathcal{N} \to \infty$, the solution to (19*a*) is $F' = 1$, except in the immediate vicinity of the wall $y = 0$. Accordingly, we set

$$F = y + f(Y) \tag{33}$$

and

$$Y = \mathcal{N}^{\frac{1}{2}} y, \tag{34}$$

† It can be seen directly from the differential equation that there is no solution for $F''(0) < 0$ (or see Leibovich 1967).

and seek an asymptotic expansion for f in the form

$$f = \sum_{1}^{\infty} \mathcal{N}^{-n+\frac{1}{2}} f_n(Y).$$

The procedure is straightforward, and yields the following expansion for the skin friction $F''(0)$,

$$F''(0) = \mathcal{N}^{\frac{1}{2}} - \tfrac{3}{4}\mathcal{N}^{-\frac{1}{2}} + O(\mathcal{N}^{-\frac{3}{2}}). \tag{35}$$

With $\mathcal{N} = 10$, (35) gives 2·924 for $F''(0)$, which differs from the exact value given in the last section by less than 1%.

It was noticed in the numerical computations that, as $\mathcal{N}$ increased, $F''(0)$ approached the mean of the bounds (31) and (32). This tendency is confirmed by (35) at least to $O(\mathcal{N}^{-\frac{3}{2}})$, since to this order (34) agrees with the mean of (31) and (32).

7. Separated flow $\mathcal{N} < 2$

For clarity, we suppose that a steady, attached flow has been established, with $\mathcal{N} \geqslant 2$. At $t = 0+$, the magnetic field strength is reduced so that $\mathcal{N}$ assumes a constant value less than 2. From our previous remarks (§§3, 4) a steady rear stagnation point flow cannot then be re-established.

Equations (9*a–e*) govern the unsteady flow, and are the magnetic counterparts of Proudman & Johnson's (1962) equation (6). Furthermore, the boundary conditions (12) apply to the unsteady case for all finite t.

Initially, F and G describe a steady R.S.P. flow as may be found from the previous sections.

For small t, the solution may be obtained by linearizing about the initial flow and then iterating. This corresponds to the classical Blasius (1908) method. For a flow which is initially potential motion, that is, started from rest, this procedure has been carried out by Tsinober *et al.* (1963).

Here, we attempt to give a description of the flow for large t, following the approach of Proudman & Johnson.

We briefly review their argument as to the rationale of looking for a solution for t large. The natural length scale in the problem at hand involves the viscosity. For any fixed finite t, however large, the thickness of the domain of flow at hand, possibly containing reversed flow and eddies, is arbitrarily small compared with the body scale as $\nu \to 0$. Since the non-dimensional time does not involve the viscosity it is concluded that effect of separation upon the external flow cannot begin (in the limit $\nu \to 0$) at finite t.

One anticipates that the length scale of the flow normal to the wall would increase continually with time under the influence of the convection field. The effect of viscosity would then diminish over most of this stretched-out flow; and it seems reasonable to assume that the situation for t large can be represented asymptotically by inviscid equations except within distances of $O(1)$ from the wall.

Accordingly, we consider ($9a, b$) with the right-hand sides set equal to zero, and follow Proudman & Johnson in seeking a similarity solution of the form†

$$F(y,t) = \lambda(t)\,f(\eta), \quad G_y(y,t) = \lambda(t)g(\eta), \quad \eta = y/\lambda(t). \tag{36}$$

Introducing ($9e$) and (36) into ($9a, b$), we find

$$(\dot\lambda/\lambda)\,(-yf'') - f'^2 + ff'' + \mathcal{N}f' - \mathcal{N} + 1 = 0,$$

$$(\dot\lambda/\lambda)\,[g - \eta g'] + fg' - f'g + \mathcal{N}(g - f) = 0.$$

A similarity solution is possible if we take

$$\lambda = e^{kt} \quad (k = \text{constant}), \tag{37}$$

and the governing equations take the form

$$(f - k\eta)f'' - f'^2 + \mathcal{N}f' - \mathcal{N} + 1 = 0, \tag{38a}$$

$$(f - k\eta)g' + (\mathcal{N} + k - f')\,g = \mathcal{N}f. \tag{38b}$$

In terms of f and g, the boundary conditions at infinity are

$$f' \to 1, \quad g' \to 1 \quad \text{as} \quad \eta \to \infty. \tag{39}$$

As Proudman & Johnson point out, in (x, η)-space, the whole of the region of (x, y)-space representing finite values of y shrinks to a thin layer on the boundary. Therefore, the kinematic condition of no normal velocity should be applied to (38), and so

$$f(0) = 0. \tag{40}$$

An immediate first integral of ($38a$) is obtainable; and, so far as f is concerned, we content ourselves with deductions which may be drawn from it. The integral is

$$A(f - k\eta) = (f' - \mathcal{N} + 1)\left(\frac{\mathcal{N} - 1 - f'}{1 - f'}\right)^{(k-1)/(2-\mathcal{N})}, \tag{41}$$

where, of course, we are specifically interested in the case $\mathcal{N} < 2$.

Since $2 - \mathcal{N} > 0$, and $f \sim \eta$ as $\eta \to \infty$, it is necessary to require $k \geqslant 1$. This restriction is identical with that imposed in the non-magnetic case. We may further note that, as in the non-magnetic case, $k = 1$ is the only value for which vorticity decays exponentially away from the boundary. In our case, there is no apparent reason that a flow with exponential decay is selected.

At $\eta = 0, f = 0$, which implies the main result of this section,

$$f'(0) = \mathcal{N} - 1, \tag{42}$$

unless $A = \infty$. The possibility $A = \infty$ may be ruled out, however, since no steady viscous solution is possible for $f'(0) \geqslant 1$ and $\mathcal{N} < 2$. Notice that a steady viscous solution is to be expected near the wall, since ultimately the boundary conditions on such a layer are steady. Therefore $A < \infty$, and (42) must hold.

When $1 < \mathcal{N} < 2$, (42) implies that fluid flows away from the wall as it leaves the viscous region. In fact (42) requires $F \sim (\mathcal{N} - 1)y$ as $y \to \infty$ in the viscous

† $f(\eta)$ and $g(\eta)$ should not be confused with the $f(y)$ and $g(y)$ which appear in the sections on steady flow.

layer. From the remarks in §3, there does exist a steady viscous solution approaching $(\mathscr{N}-1)y$ as $y \to \infty$.

It would therefore appear that, when $1 < \mathscr{N} < 2$, a phenomenon which could be labelled 'separation' occurs in the Proudman–Johnson framework, *without reversed flow.*

In fact, reversed flow would only occur with $\mathscr{N} < 1$. (When $\mathscr{N} = 1$, the skin friction at the wall vanishes.) In this event, (42) shows that there is an inviscid flow towards the wall as it is approached, and the picture is similar to the non-magnetic one, to which it reduces when $\mathscr{N} = 0$. The inviscid domain is one of growing eddies which perform the transition between a forward stagnation point flow near the wall, and a R.S.P. flow far away. The question of whether there is a steady forward stagnation point solution to our equations is answered in the affirmative by, for example, Ludford (1963).

A solution for g satisfying the first-order linear equation (38*b*) and the boundary condition (39) can be easily written down in terms of f if desired.

8. Discussion

(*a*) *Steady R.S.P. flow*

In §§3–6, it was shown that boundary-layer separation does not begin at the rear stagnation point if $\mathscr{N} > 2$.

A simple physical argument can be advanced to show why a R.S.P. flow may be possible for $\mathscr{N} \neq 0$, when none exists for $\mathscr{N} = 0$.† A fluid particle acquires rotation as it enters a viscous boundary layer. Consider those particles which travel near the 'outer edge' of the layer, i.e. at large values of the boundary-layer co-ordinate y. If R.S.P. flow exists, then this fluid must be the first to pass out of the viscous region at the R.S.P. In typical boundary layer flows, however, u_{yyy} and u_y are of the same sign for large y; hence, the viscous forces produce vorticity at the 'outer edge', which is always in the same direction. Thus, the fluid travelling in the viscous region, but far from the wall, has its vorticity increased. But, to leave the boundary layer, the fluid must lose its vorticity, and, as we have seen, for $\mathscr{N} = 0$, the only mechanism available, viscosity, does not serve this purpose. Therefore fluid cannot pass from the viscous layer into the irrotational external flow, and it thus appears that R.S.P. flow will not occur.

For $\mathscr{N} \neq 0$, the external inviscid stream may have electromagnetically acquired vorticity. Nevertheless, fluid particles must lose excess vorticity generated by viscous effects before entry into the inviscid stream is possible. It is well known (e.g. Shercliff 1965) that magnetic fields can provide a means for damping vorticity.

This effect is easily seen in our case. If we take the simplest case, $\theta = \frac{1}{2}\pi$, as an example, the Helmholz equation for the single (z) component of vorticity (Ω) takes the form

$$\frac{D\Omega}{Dt} = -\mathscr{N}\Omega + \frac{\partial^2 \Omega}{\partial y^2}.$$

† The author is indebted to Professor F. K. Moore for providing this explanation of the non-existence of R.S.P. flow in the non-magnetic case.

If the viscous term here is ignored, it is evident that vorticity decays exponentially with distance traversed along a streamline, with a relaxation time proportional to $\mathcal{N}$. Hence vorticity is created at the wall and diffused into the fluid by viscous forces, and is destroyed by interaction with the magnetic field.

Apparently, when $\mathcal{N} > 2$, the magnetic field is capable of damping all of the viscosity-generated vorticity before a particle passes out of the viscous region. The physical significance of the particular value $\mathcal{N} = 2$, however, is still obscure.

(*b*) *A critique of the experiment on MHD boundary-layer separation*

It should be pointed out that there is some doubt about exactly what the experimental data reported by Tsinober *et al.* represents. Neither Moreau's work nor that presented here indicates a dependence on the body conductivity such as is shown by the experimental data.

One obvious reason for this is due to the fact that, in the analysis, $\mathcal{N}$ is based on the magnetic field which obtains at the solid boundary, and the experimental findings reported are based upon the applied field. Conductivity of the cylinders can affect the overall distribution of magnetic field, especially when the fluid and body are good conductors.

In fact, the applied field in the experiment was transverse to the undisturbed stream, and presumably the normal component of magnetic field vanished at the R.S.P. As explained below, there are reasons to suspect that N, based on the applied field, was actually considerably larger than 2. In this event, $\mathcal{N} > 2$ over a large portion of the rear of the cylinder, and this may be sufficient to prevent separation from being observed. Of course, attached flow at the R.S.P. is not actually possible for $\mathcal{N} < 2$, as we have shown.

Furthermore, the experiment utilizes the fact that tin and bismuth, from which the cylinders were manufactured, dissolve in mercury. The separation line, which divides the laminar upstream flow from the eddying separated flow, is apparently thereby made visible. The values of N for which separation was reported inhibited were based on the properties of mercury, although it is evident from the analysis here that N should be based upon the properties of the fluid adjacent to the solid. Presumably in the experiment, that fluid is an amalgam of mercury and either tin or bismuth, with properties which are not known.

(*c*) *Unsteady flow at the R.S.P.*

If the Proudman–Johnson model is appropriate as $t \to \infty$, then separation occurs with positive skin friction and without reverse flow when $1 < \mathcal{N} < 2$.

Reversed flow and negative skin friction occur for $\mathcal{N} < 1$, and the results of Proudman & Johnson are recovered for $\mathcal{N} = 0$.

Since separation has always been associated with the onset of reversed flow, and since the magnetic field clearly provides a means of controlling the boundary layer, the results obtained here suggest that magnetic fields may be used to advantage in experiments designed to study the fundamentals of the separation process.

The assistance of Miss Sheila Burroughs with the numerical computations is acknowledged with thanks. Prof. K. Stewartson read a first draft of this paper, and his helpful and penetrating criticisms are in large part responsible for a broadening of its scope. This work was begun while the author held a N.A.T.O. Postdoctoral Fellowship at University College London.

REFERENCES

BLASIUS, H. 1908 Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.* **56**, 1.

BROWN, S. N. & STEWARTSON, K. 1965 On similarity solutions of the boundary-layer equations with algebraic decay. *J. Fluid Mech.* **23**, 673–87.

COPPEL, W. A. 1960 On a differential equation of boundary-layer theory. *Phil. Trans. Roy. Soc.* A **253**, 101–35.

FUCKS, W., FISCHER, E. & UHLENBUSCH, J. 1964 Laminare Grenzschichten an zylindrischen Körpern in der Magnetohydrodynamik. *Z. Flugwissenschaften*, **11**, 389–95.

GLAUERT, M. B. 1956 The laminar boundary-layer on oscillating plates and cylinders. *J. Fluid Mech.* **1**, 97–110.

GOLDSTEIN, S. 1965 On backwards boundary layers and flow in converging passages. *J. Fluid Mech.* **21**, 33–45.

HARDY, G. H. 1939 *Proc. Camb. Phil. Soc.* **35**, 338–40.

HEISER, W. H. & SHERCLIFF, J. A. 1965 A simple demonstration of the Hartmann layer. *J. Fluid Mech.* **22**, 701–7.

LEIBOVICH, S. 1967 On the differential equation for 'backward' boundary-layers with a magnetic field. To appear in *Proc. Camb. Phil. Soc.*

LUDFORD, G. S. S. 1963 Hydromagnetic stagnation-point flow for small R_M. *Z. angew. Math. Mech.* **43**, 9–24.

MOREAU, R. 1964 L'effet d'un champ magnétique transversal sur le decollement. *C. r. hebd. Séanc. Acad. Sci., Paris* **258**, 1732–35.

PROUDMAN, I. & JOHNSON, K. 1962 Boundary-layer growth near a rear stagnation point. *J. Fluid Mech.* **12**, 161–8.

SEARS, W. R. 1961 On a boundary-layer phenomenon in magneto-fluid dynamics. *Astro. Acta* **7**, 223–36.

SHERCLIFF, J. A. 1965 *A Textbook of Magnetohydrodynamics*, pp. 84–6. Oxford: Pergamon Press.

TSINOBER, A., SHTERN, A. & SHCHERBININ, E. 1963 On the separation of the magnetohydrodynamic boundary-layer [in Russian]. *Latvijas PSR Zinatnu Akad. Vestis* **12**, 49–56. Also [unedited English translation] Foreign Technology Division, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, U.S.A., FTD–TT–65–27.